Problem 02: Greatest Common Denominator

Given two natural numbers x and y, compute the greatest common denominator.

$$
\begin{array}{rcl}
A & = & \underset{x}{\mathbb{N}} \times \underset{y}{\mathbb{N}} \times \underset{z}{\mathbb{Z}} \\
B & = & \underset{x'}{\mathbb{N}} \times \underset{y'}{\mathbb{N}} \\
Q & = & (x' = x) \land (y' = y) \\
R & = & Q \land (z|x) \land (z|y) \land \forall k \in [z+1, \min(x,y)] : (k \nmid x \lor k \nmid y)
\end{array}
$$

Solution

We iterate z from $\min(x, y)$ to 1 and look for the first common denominator, which (since we start from the "top") will be the greatest one:

$$
\begin{array}{rcl}
P & = & Q \land z \in [1, \min(x,y)] \land \forall k \in [z+1, \min(x,y)] : (k \nmid x \lor k \nmid y) \\
\neg\pi & = & (z = 1) \lor (z|x \land z|y) \\
\pi & = & (z \neq 1) \land (z \nmid x \lor z \nmid y) \\
t & = & z \\
Q' & = & Q \land (z = \min(x,y))
\end{array}
$$

Solving this for $z \leftarrow (z-1)$ we can see that $P \land \pi \Rightarrow \mathrm{wp}((z := z-1), P)$, thus, nothing more is required in the body of the loop:

$$
\begin{array}{rcl}
P^{z \leftarrow z-1} & = & Q \land (z-1) \in [1, \min(x,y)] \land \forall k \in [z, \min(x,y)] : (k \nmid x \lor k \nmid y) \\
& \simeq & P \land \pi
\end{array}
$$

The resulting program:

$z := \min(x, y)$
while $(z \neq 1) \land (z \nmid x \lor z \nmid y)$
$\quad z := z - 1$